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BULK VISCOUS SOLUTIONS TO THE FIELD EQUATIONS AND THE DECELERATION PARAMETER-REVISITED

ANIRUDH PRADHAN,* IOTEMSHI I.

Department of Mathematics, Hindu Post-graduate College,
Zamania, Ghazipur 232 331, India

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We utilise a form for the Hubble parameter to generate a number of solutions to the Einstein field equations with variable cosmological constant and variable gravitational constant in the presence of a bulk viscous fluid. The Hubble law utilised yields a constant value for the deceleration parameter. A new class of solutions is presented in the Robertson-Walker spacetimes. The coefficient of bulk viscosity is assumed to be a power function of the mass density. For a class of solutions, the deceleration parameter is negative which is consistent with the supernovae Ia observations.

1. Introduction

Einstein proposed his General Theory of Relativity as a geometric theory. He introduced a cosmological constant into his field equations in order to obtain a static cosmological model as without the cosmological term, the field equations would admit only non-static solutions. The Einstein field equations are a coupled system of highly nonlinear differential equations and we seek physical solutions for applications in cosmology and astrophysics. In order to solve the field equations we normally assume a form for the matter content or suppose that spacetime admits Killing vector symmetries [1]. Solutions to the field equations may also be generated by applying a law of variation for Hubble's parameter. It is interesting to observe that this law yields a constant value for the deceleration parameter. The variation of Hubble's law assumed is not inconsistent with observation and has the advantage of providing simple functional forms of the scale factor. In the simplest case the Hubble law yields a constant value for the deceleration parameter. In earlier literature cosmological models with a constant deceleration parameter have been studied by Berman [2,3], Berman and Gomide [4], Johri and Desikan [5], Singh and Desikan [6], Pradhan et al [7] and others.

Models with a relic cosmological constant Λ have received considerable attention recently among researchers for various reasons (see Refs. [8−12] and references therein). Some

*E-mail: acpradhan@yahoo.com, pradhan@iucaa.ernet.in (Corresponding Author)

of the recent discussions on the cosmological constant “problem” and on cosmology with a time-varying cosmological constant by Ratra and Peebles [13], Dolgov [14−16] and Sahni and Starobinsky [17] point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a “constant”, however, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress-energy with a time-varying Λ can be found. For these solutions, conservation of energy requires decrease in the energy density of the vacuum component to be compensated by a corresponding increase in the energy density of matter or radiation. Earlier researchers on this topic, are contained in Zeldovich [18], Weinberg [9] and Carroll, Press and Turner [19]. Recent observations by Perlmutter *et al* [20] and Riess *et al* [21] strongly favour a significant and positive Λ. Their finding arise from the study of more than 50 type Ia supernovae with redshifts in the range $0.10 \leq z \leq 0.83$ and suggest Friedmann models with negative pressure matter such as a cosmological constant, domain walls or cosmic strings (Vilenkin [22], Garnavich *et al* [23]. The main conclusion of these works is that the expansion of the universe is accelerating.

Several ans$\ddot{a}$tz have been proposed in which the Λ term decays with time (see Refs. Gasperini [24,25], Freese *et al* [26], $\ddot{O}$zer and Taha [12], Peebles and Ratra [27], Chen and Hu [28], Abdussattar and Viswakarma [29], Gariel and Le Denmat [30], Pradhan *et al* [31]). Of the special interest is the ansatz $\Lambda \propto S^{-2}$ (where S is the scale factor of the Robertson-Walker metric) by Chen and Wu [28], which has been considered/modified by several authors (Abdel-Rahaman [32], Carvalho *et al* [12], Waga [33], Silveira and Waga [34], Vishwakarma [35]).

In most treatments of cosmology, cosmic fluid is considered as perfect fluid. However, bulk viscosity is expected to play an important role at certain stages of expanding universe [36−38]. It has been shown that bulk viscosity leads to inflationary like solution [39], and acts like a negative energy field in an expanding universe [40]. A number of authors have discussed cosmological solutions with bulk viscosity in various context [41−44].

It has been shown by Berman [2] and Berman & Gomide [4] that all the phases of the universe, i.e., radiation, inflation and pressure-free, may be considered as particular cases of the deceleration parameter q = constant type, as

$$q = -\frac{S\ddot{S}}{\dot{S}^2},$$

where dots stand for time derivatives. We extend this definition to the Robertson-Walker cosmological models. In the past few decades there have been numerous modification of general relativity in which gravitational constant G varies with time [45]. Considering the principle of absolute quark confinement, Der Sarkissian [46] has suggested that gravitational and cosmological constants may be considered as functions of time parameter in Einstein’s theory of relativity. A number of authors [47−50] have considered time-varying G and Λ within the frame work of general relativity.

Maharaj and Naidoo [51] utilised a form of the Hubble parameter to generate a number of solutions to the Einstein field equations with variable cosmological constant and variable gravitational constant in presence of a perfect fluid as the source of matter. These authors have extended the results obtained by Berman [2,3], Berman and Gomide [4] by obtaining solutions to Einstein field equations, with variable gravitational and cosmological constants, in the Robertson-Walker spacetime. Explicit forms for the gravitational constant, cosmological constant, scale factor, energy density and pressure are obtained for various cases.

Motivated by the situations discussed above that bulk viscosity, gravitational and cosmological "constants", are more relevant during early stages of the universe, our intension in this paper is to extend the results obtained by Berman [3], Maharaj and Naidoo [51] by including a bulk viscous fluid as a source of matter in the energy momentum tensor. This paper is organized as follows. In section 2, the solutions and main results of Berman [3], Maharaj and Naidoo [51] are reviewed. In section 3, the corresponding solutions for a universe filled with a bulk viscous fluid are found. In section 4, a number of classes of new solutions for all cases of $k : 0, 1, -1$ for variable G and Λ are presented. Our results are discussed in section 5.

2. Robertson-Walker Spacetime-Revisited

In the standard coordinates (x^a) = (t, r, θ, ϕ) the Robertson-Walker line element has the form

$$ds^2 = -dt^2 + S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta d\phi^2)\right], \tag{1}$$

where S(t) is a cosmic scale factor. Without loss of generality the constant k is related to the spatial geometry of a 3-dimensional manifold generated by t = constant. The Robertson-Walker spacetimes are the standard cosmological models and are consistent with observational results. For the case of variable cosmological constant $\Lambda(t)$ and gravitational constant $G(t)$ the Einstein field equations

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab} \tag{2}$$

yield

$$\frac{3}{S^2}(\dot{S}^2 + k) = 8\pi G\mu + \Lambda \tag{3}$$

$$2\frac{\ddot{S}}{S} + \frac{(\dot{S}^2 + k)}{S^2} = -8\pi G p + \Lambda \tag{4}$$

for the line element (1). From Eqs. (3) and (4) we obtain the generalised continuity equation

$$\dot{\mu} + 3\frac{\dot{S}}{S}(\mu + p) + \frac{\dot{G}}{G}\mu + \frac{\dot{\Lambda}}{8\pi G} = 0. \tag{5}$$

This reduces to the conventional continuity equation for constant Λ and G. If the classical conservation law, $T^{ab}_{;b} = 0$, also holds, then Eq. (5) implies two relationships

$$\dot{\mu} + 3\frac{\dot{S}}{S}(\mu + p) = 0, \tag{6}$$

$$8\pi\mu\dot{G} + \dot{\Lambda} = 0, \tag{7}$$

which facilitate the solutions of the field equations. The result (6) is just the conventional continuity equation, and (7) simply relates G and Λ and does not explicitly contain the scale factor $S(t)$.

Maharaj and Naidoo [51] considered the generalised Einstein field equations (3)-(4)with variable gravitational constant G and variable cosmological constant $\Lambda(t)$ for the Robertson-Walker metric (1). They assumed the variation of the Hubble parameter as given by the equation

$$H \equiv \frac{\dot{S}}{S} = DS^{-m} \tag{8}$$

where D and m are constants. Then this imply that the deceleration parameter q is constant i. e. $q = m - 1$.

The form of the Hubble parameter (8) was first utilised by Berman [2] and Berman and Gomide [4] for the case of classical Einstein field equations with $\dot{\Lambda} = 0$ and $\dot{G} = 0$. Berman [3] presented a solution to the field equations (2) for the $k = 0$ Robertson-Walker spacetime:

$$S = (C + mDt)^{1/m}, \tag{9}$$

$$\Lambda = BS^{-2m}, \tag{10}$$

$$G = \beta S^{mB/(4\pi A)}, \tag{11}$$

$$\mu = \frac{A}{\beta}S^{-2m-mB/(4\pi A)}, \tag{12}$$

$$p = \frac{A}{3\beta}\left[m(2 + \frac{B}{4\pi A}) - 3\right]S^{-2m-mB/(4\pi A)}, \tag{13}$$

where A, B, C, β are constants and are subject to the following condition

$$3D^2 = 8\pi A + B$$

It is worth noting that Eq. (16) given by Berman [3] corresponding to Maharaj and Naidoo [51] equation (13), has an incorrect coefficient on the right hand side. Equations (9)-(13) comprise the general solution to the generalised Einstein field equations (3)-(4) with variable G and Λ for the Hubble law (8).

3. Bulk Viscous Solutions of the Field Equations

In this section bulk viscous models of the universe are discussed. Weinberge [9] has suggested that in order to consider the effect of bulk viscosity, the perfect fluid pressure should be replaced by the effective pressure $\bar{p}$ by

$$\bar{p} = p - \xi\theta, \tag{14}$$

where ξ is the coefficient of bulk viscosity and θ is the expansion scalar given by $\theta = 3H$. Here ξ is, in general, a function of time.
Therefore, from Eq. (13), we obtain

$$p - \xi\theta = \frac{A}{3\beta}\left[m(2 + \frac{B}{4\pi A}) - 3\right] S^{-2m - mB/(4\pi A)} \tag{15}$$

For complete determinacy of the system, we assume an equation of state of an ideal gas given by

$$p = \gamma\mu, \quad 0 \leq \gamma \leq 1, \tag{16}$$

where γ is a constant. Thus, given $\xi(t)$ we can solve for the cosmological parameters. It is standard to assume [42,52] the following widely accepted *ad hoc* law

$$\xi(t) = \xi_0 \mu^n \tag{17}$$

If $n = 1$, Eq. (17) may correspond to a radiating fluid, whereas $n = 3/2$ may correspond to a string-dominated universe [53]. However, more realistic models [54] are based on n lying the regime $0 \leq n \leq 1/2$. On using Eq. (17) in Eq. (15), we obtain

$$p - \xi_0\mu^n\theta = \frac{A}{3\beta}\left[m(2 + \frac{B}{4\pi A}) - 3\right] S^{-2m - mB/(4\pi A)} \tag{18}$$

3.1. *Model I:* $(\xi = \xi_0)$

When $n = 0$, Eq. (17) reduces to $\xi = \xi_0$ = constant and hence Eq. (18) with Eq. (16) gives

$$\mu = \frac{S^{-m}}{\gamma}\left[3D\xi_0 + \frac{A}{3\beta}\left[m(2 + \frac{B}{4\pi A}) - 3\right]\right] S^{-2m - mB/(4\pi A)} \tag{19}$$

3.2. *Model II:* $(\xi = \xi_0\mu)$

When $n = 1$, Eq. (17) reduces to $\xi = \xi_0\mu$ and hence Eq. (18) with Eq. (16) gives

$$\mu = \frac{A}{3\beta(\gamma - 3D\xi_0 S^{-m})}\left[m(2 + \frac{B}{4\pi A}) - 3\right] S^{-2m - mB/(4\pi A)} \tag{20}$$

It is possible to avoid the horizon and monopole problem with the above variable $G(t)$ and $\Lambda(t)$ solutions as suggested by Berman [3]. Other models are also considered by Berman *et al* [55] and Bertolami [47] which have the relationship

$$\Lambda \propto \frac{1}{t^2}.$$

This form of Λ is physically reasonable as observations suggest that Λ is very small in the present universe. A decreasing functional form permits Λ to be large in the early universe. A partial list of cosmological models in which the gravitational constant G is decreasing function of time are contained in Grøn [41], Hellings *et al* [56], Rowan-Robinson [57], Shapiro *et al* [58] and Van Flandern [59]. The possibility of G increasing with time, at least in some stages of the development of the universe, has been investigated by Abdel-Rahman [32], Chow [60], Levitt [61] and Milne [62]. In these models, by selecting proper signs of different constants, one can make μ positive and decreasing function of time t.

4. Other Solutions-Revisited

Maharaj and Naidoo [51] presented a number of classes of new solutions for all classes of $k : 0, 1, -1$ for variable cosmological constant Λ and variable gravitational constant G for Hubble law (8). These solutions covered both the cases of $m = 0$ and $m \neq 0$ for the scale factor S:

$$S(t) = \left[\begin{array}{ll} [C + mDt]^{1/m} & \text{when } m \neq 0 \\ Ee^{Dt} & \text{when } m = 0 \end{array} \right. \tag{21}$$

where C, D, E are constants. To solve the Einstein field equations (3)-(4) they adopted the ansatz

$$\frac{3D^2}{S^{2m}} - \Lambda = K, \tag{22}$$

$$8\pi G\mu - \frac{3k}{S^2} = K \tag{23}$$

where K is constant. This ansatz has the advantage of providing further classes of solutions. Maharaj and Naidoo [51] obtained the expressions for cosmological constant Λ, energy density μ in terms of the gravitational constant G and the scale factor S

$$\Lambda = \frac{3D^2}{S^{2m}} - K \tag{24}$$

$$\mu = \frac{1}{8\pi G}\left[\frac{3k}{S^2} + K\right] \tag{25}$$

On substituting Eq. (25) and the derivative of Eq. (24) with respect to time coordinate t into Eq. (7) we obtain the differential equation

$$\frac{\dot{G}}{G} = 6mD^2\frac{\dot{S}}{S^{2m+1}}\frac{1}{(\frac{3k}{S^2} + K)} \tag{26}$$

relating G to S. Thus the scalar factor S is specified by our assumed form of the Hubble parameter, the gravitational constant G is known in principle. The ansatz (22)-(23) enables to integrate all the Einstein field equations for a number of values of m, k and K. In the remainder of this section we present a variety of classes of solutions to the Einstein field equations for each of these cases considered in the presence of bulk viscosity. We list the form of the scale factor S, the variable cosmological constant Λ, the variable gravitational constant G, the energy density μ and the effective pressure $\bar{p}$. There are other classes of solution possible for other values of m. However the integration becomes extremely complicated for general m, we present only some simple cases in the following.

4.1. *Case I :* $m = 0, K \neq 0, k \neq 0$.

In this case we obtain

$$S = Ee^{Dt}, \tag{27}$$

$$\Lambda = 3D^2 - K, \tag{28}$$

$$G = A, \tag{29}$$

$$\mu = \frac{1}{8\pi A}\left[\frac{3k}{S^2} + K\right], \tag{30}$$

$$\bar{p} = -\frac{1}{8\pi A}\left[\frac{k}{S^2} + K\right]. \tag{31}$$

4.1.1. *Model I :* $(\xi = \xi_0)$

When $n = 0$, Eq. (17) reduces to $\xi = \xi_0$ = constant and hence Eq. (31) with Eq. (16) gives

$$\mu = \frac{1}{\gamma}\left[3\xi_0 D - \frac{1}{8\pi A}\left[\frac{k}{S^2} + K\right]\right]. \tag{32}$$

In this model, if we set $A < 0$ and $\xi_0, D, K, k > 0$ then μ is always positive and decreasing function with time.

4.1.2. *Model II :* $(\xi = \xi_0 \mu)$

When $n = 1$, Eq. (17) reduces to $\xi = \xi_0 \mu$ and hence Eq.(31) with Eq. (16 gives

$$\mu = -\frac{1}{8\pi A(\gamma - 3D\xi_0)}\left[\frac{k}{S^2} + K\right]. \tag{33}$$

In these de Sitter-type solutions Λ and G are strictly constant because of the restriction $m = 0$. The cosmological constant Λ vanishes when $K = 3D^2$ and is positive for $K < 3D^2$. The scale factor S is exponential in t, so that if $D > 0$ then the universe is exponentially expanding always. Such models are not physical description of our present universe but could be applicable in the early universe in the inflationary scenario. For $m = 0$, we get the deceleration parameter $q = -1$ for these class of solutions which is consistent with the recent observations of supernovae Ia which require that the present universe is accelerating [20,21]. In model II, if we set $A < 0$, and $\xi_0, D, K, k > 0$ and $\gamma > 3D\xi_0$, then μ is always positive and decreasing function with time.

4.2. *Case II* : $m \neq 0, K = 0, k \neq 0$.

In this case we obtain

$$S = [C + mDt]^{1/m}, \tag{34}$$

$$\Lambda = \frac{3D^2}{S^{2m}}, \tag{35}$$

$$G = \alpha\, exp\{\frac{mD^2}{k(1-m)} S^{2-2m}\}, \tag{36}$$

$$\mu = \frac{3k}{8\pi\alpha} S^{-2} exp\{\frac{mD^2}{k(1-m)} S^{2-2m}\}, \tag{37}$$

$$\bar{p} = -\frac{1}{8\pi\alpha}\left[\frac{4mD^3}{S^{3m}} + \frac{3k}{S^2}\right] exp\{\frac{mD^2}{k(1-m)} S^{2-2m}\}. \tag{38}$$

4.2.1. *Model I* : $(\xi = \xi_0)$

When $n = 0$, Eq. (17) reduces to $\xi = \xi_0$ = constant and hence Eq. (38) with Eq. (16) gives

$$\mu = \frac{1}{\gamma}\left[3D\xi_0 S^{-m} - \frac{1}{8\pi\alpha}\left[\frac{4mD^3}{S^{3m}} + \frac{3k}{S^2}\right] exp\{\frac{mD^2}{k(1-m)} S^{2-2m}\}\right]. \tag{39}$$

4.2.2. *Model II* : $(\xi = \xi_0 \mu)$

When $n = 1$, Eq. (17) reduces to $\xi = \xi_0 \mu$ and hence Eq.(38) with Eq. (16) gives

$$\mu = -\frac{1}{8\pi\alpha(\gamma - 3D\xi_0 S^{-m})}\left[\frac{4mD^3}{S^{3m}} + \frac{3k}{S^2}\right] exp\{\frac{mD^2}{k(1-m)} S^{2-2m}\}. \tag{40}$$

This case shares the common feature that G may be increasing in time in certain regions of spacetime with the model proposed by Abdel-Rahman [32].

4.3. *Case III* : $m \neq 0, K \neq 0, k = 0$.

In this case we obtain

$$S = [C + mDt]^{1/m}, \tag{41}$$

$$\Lambda = \frac{3D^2}{S^{2m}} - K, \tag{42}$$

$$G = \alpha\ \ exp\{\frac{3D^2}{KS^{2m}}\}, \tag{43}$$

$$\mu = \frac{K}{8\pi\alpha} exp\{-\frac{3D^2}{KS^{2m}}\}, \tag{44}$$

$$\bar{p} = -\frac{1}{8\pi\alpha}\left[\frac{2mD^2}{S^{2m}} + K\right] exp\{-\frac{3D^2}{KS^{2m}}\}. \tag{45}$$

4.3.1. *Model I* : $(\xi = \xi_0)$

When $n = 0$, Eq. (17) reduces to $\xi = \xi_0$ = constant and hence Eq. (45) with Eq. (16) gives

$$\mu = \frac{1}{\gamma}\left[3D\xi_0 S^{-m} - \frac{1}{8\pi\alpha}\left[\frac{2mD^3}{S^{2m}} + K\right] exp\{-\frac{3D^2}{KS^{2m}}\}\right]. \tag{46}$$

4.3.2. *Model II* : $(\xi = \xi_0\mu)$

When n = 1, Eq. (17) reduces to $\xi = \xi_0\mu$ and hence Eq. (45) with Eq. (16) gives

$$\mu = -\frac{1}{8\pi\alpha(\gamma - 3D\xi_0 S^{-m})}\left[\frac{2mD^2}{S^{2m}} + K\right] exp\{\frac{-3D^2}{KS^{2m}}\}. \tag{47}$$

4.4. *Case IV* : $m = 2, K \neq 0, k \neq 0$.

In this case we obtain

$$S = [C + 2Dt]^{1/2}, \tag{48}$$

$$\Lambda = \frac{3D^2}{S^4} - K, \tag{49}$$

$$G = \alpha\left[\frac{(3kS^{-2} + K)^{K/k}}{exp\{S^{-2}\}}\right]^{2D^2/k}, \tag{50}$$

$$\mu = \frac{1}{8\pi\alpha}[3kS^{-2} + K]\left[\frac{exp\{S^{-2}\}}{(3kS^{-2} + K)^{K/k}}\right]^{2D^2/k}, \tag{51}$$

$$\bar{p} = -\frac{1}{8\pi\alpha}[3kS^{-2} + K]\left[\frac{exp\{S^{-2}\}}{(3kS^{-2} + K)^{K/k}}\right]^{2D^2/k} \times$$
$$\left[\frac{4D^2S^{-4}(2K - 3kS^{-2})}{k(3kS^{-2} + K)} + 1\right] + \frac{kS^{-2}}{4\pi\alpha}\left[\frac{exp\{S^{-2}\}}{(3kS^{-2} + K)^{K/k}}\right]^{2D^2/k}. \tag{52}$$

Unlike the cases considered thus far we have a specific value for m. This gives a value $q = 1$ for the deceleration parameter. A wide range of behaviour is possible for the gravitational constant.

4.4.1. *Model I* : $(\xi = \xi_0)$

When $n = 0$, Eq. (17) reduces to $\xi = \xi_0$ = constant and hence Eq. (52) with Eq. (16) gives

$$\mu = \frac{3D\xi_0 S^{-2}}{\gamma} - \frac{1}{8\pi\alpha\gamma}[3kS^{-2} + k]\left[\frac{exp\{S^{-2}\}}{(3kS^{-2} + K)^{K/k}}\right]^{2D^2/k} \times$$
$$\left[\frac{4D^2S^{-4}(2K - 3kS^{-2})}{k(3kS^{-2} + K)} + 1 - \frac{2kS^{-2}}{(3kS^{-2} + K)}\right]. \tag{53}$$

4.4.2. *Model II :* $(\xi = \xi_0\mu)$

When $n = 1$, Eq. (17) reduces to $\xi = \xi_0\mu$ and hence Eq. (52) with Eq. (16) gives

$$\mu = -\frac{1}{8\pi\alpha(\gamma - 3D\xi_0 S^{-2})}[3kS^{-2} + K]\left[\frac{exp\{S^{-2}\}}{(3kS^{-2}+K)^{K/k}}\right]^{2D^2/k} \times$$
$$\left[\frac{4D^2S^{-4}(2K - 3kS^{-2})}{k(3kS^{-2}+K)} + 1 - \frac{2kS^{-2}}{(3kS^{-2}+K)}\right]. \tag{54}$$

4.5. *Case V :* $m = -2, K \neq 0, k \neq 0$.

In this case we obtain

$$S = \frac{1}{\sqrt{C - 2Dt}}, \tag{55}$$

$$\Lambda = \frac{3D^2}{S^{-4}} - K, \tag{56}$$

$$G = \alpha\frac{exp\{\frac{3kS^2}{K^2} - \frac{S^4}{2K}\}}{[S^{-2}(3kS^{-2}+K)]^{9k^2/K^3}}, \tag{57}$$

$$\mu = \frac{S^{-18k^2/K^3}}{8\pi\alpha}[3kS^{-2}+K]^{9k^2/K^3+1}exp\{\frac{S^4}{2K} - \frac{3kS^2}{K^2}\}, \tag{58}$$

$$\bar{p} = \frac{S^{-18k^2/K^3}}{4\pi\alpha}[3kS^{-2}+K]^{9k^2/K^3+1}\left[\frac{6k^2}{K^3} + \frac{3k - S^6}{3S^2(3kS^{-2}+K)} - \frac{1}{2}\right] \times$$
$$exp\{\frac{S^4}{2K} - \frac{3kS^2}{K^2}\}. \tag{59}$$

4.5.1. *Model I :* $(\xi = \xi_0)$

When $n = 0$, Eq. (17) reduces to $\xi = \xi_0$ = constant and hence Eq. (59) with Eq. (16) gives

$$\mu = \frac{3D\xi_0 S^2}{\gamma} + \frac{S^{-18k^2/K^3}}{4\pi\alpha\gamma}\left[3kS^{-2}+K\right]^{9k^2/K^3+1} \times$$
$$\left[\frac{6k^2}{K^3} + \frac{3k - S^6}{3S^2(3kS^{-2}+K)} - \frac{1}{2}\right]exp\{\frac{S^4}{2K} - \frac{3kS^2}{K^2}\}. \tag{60}$$

4.5.2. *Model II :* $(\xi = \xi_0\mu)$

When $n = 1$, Eq (17) reduces to $\xi = \xi_0\mu$ and hence Eq. (59) with Eq. (16) gives

$$\mu = \frac{S^{-18k^2/K^3}}{4\pi\alpha(\gamma - 3D\xi_0 S^2)}[3kS^{-2}+K]^{9k^2/K^3+1} \times$$
$$\left[\frac{6k^2}{K^3} + \frac{3k - S^6}{3S^2(3kS^{-2}+K)} - \frac{1}{2}\right]exp\{\frac{S^4}{2K} - \frac{3kS^2}{K^2}\}. \tag{61}$$

For $m = -2$, we get the deceleration parameter $q = -3$ for these class of solutions which is consistent with the recent observations of supernovae Ia which require the present universe is accelerating [20,21]

4.6. *Case VI :* $m = \frac{1}{2}, K \neq 0, k \neq 0$.

In this case we obtain

$$S = [C + \frac{1}{2}Dt]^2, \tag{62}$$

$$\Lambda = \frac{3D^2}{S} - K, \tag{63}$$

$$G = \alpha\, exp\{\sqrt{\frac{3}{kK}}D^2 arctan(\sqrt{\frac{K}{3k}}S)\}, \tag{64}$$

$$\mu = \frac{1}{8\pi\alpha}\left[3kS^{-2} + K\right] exp\{-\sqrt{\frac{3}{kK}}D^2 arctan(\sqrt{\frac{K}{3k}}S)\}, \tag{65}$$

$$\bar{p} = -\frac{1}{8\pi\alpha}\left[(k - D^2S)S^{-2} + K\right] exp\{-\sqrt{\frac{3}{kK}}D^2 arctan(\sqrt{\frac{K}{3k}}S)\}. \tag{66}$$

For this class of solutions the deceleration parameter has the value $q = -\frac{1}{2}$ as $m = \frac{1}{2}$.

4.6.1. *Model I :* $(\xi = \xi_0)$

When $n = 0$, Eq. (17) reduces to $\xi = \xi_0$ = constant and hence Eq. (66) with Eq. (16) gives

$$\mu = \frac{3D\xi_0}{\gamma S^{1/2}} - \frac{1}{8\pi\alpha}\left[(k - D^2S)S^{-2} + K\right] \times$$

$$exp\{-\sqrt{\frac{3}{kK}}D^2 arctan(\sqrt{\frac{K}{3k}}S)\}. \tag{67}$$

4.6.2. *Model II :* $(\xi = \xi_0\mu)$

When $n = 1$, Eq. (17) reduces to $\xi = \xi_0\mu$ and hence Eq. (66)with Eq. (16) gives

$$\mu = -\frac{1}{8\pi\alpha(\gamma - 3D\xi_0 S^{-1/2})}\left[(k - D^2S)S^{-2} + K\right] \times$$

$$exp\{-\sqrt{\frac{3}{kK}}D^2 arctan(\sqrt{\frac{K}{3k}}S)\}. \tag{68}$$

For these class of solutions, it is observed that the universe is accelerating.

4.7. *Case VII :* $m = \frac{2}{3}, K \neq 0, k \neq 0$

In this case we obtain

$$S = [C + \frac{2}{3}Dt]^{3/2}, \tag{69}$$

$$\Lambda = \frac{3D^2}{S^{4/3}} - K, \tag{70}$$

$$G = \alpha \left[\frac{(S^{2/3}+a)^2}{S^{4/3} - aS^{2/3} + a^2} exp\{6\ arctan(\frac{2S^{2/3}-a}{\sqrt{3}a})\}\right]^{D^2/Ka^2}, \tag{71}$$

$$\mu = \frac{1}{8\pi\alpha}[3kS^{-2} + K]\left[\frac{S^{4/3} - aS^{2/3} + a^2}{(S^{2/3}+a)^2} exp\{-6\ arctan(\frac{2S^{2/3}-a}{\sqrt{3}a})\}\right]^{D^2/Ka^2}, \tag{72}$$

$$\bar{p} = -\frac{1}{8\pi\alpha}\left[\frac{S^{4/3} - aS^{2/3} + a^2}{(S^{2/3}+a)^2} exp\{-6\ arctan(\frac{2S^{2/3}-a}{\sqrt{3}a})\}\right]^{D^2/Ka^2} \times \left[kS^{-2} + K + \frac{2D^2}{3Ka}(3kS^{-2}+K)\left[\frac{(S^{2/3}+a)^2}{S^{4/3} - aS^{2/3} + a^2}\right]\right] \times \left[\frac{S^{2/3}-a}{(S^{2/3}+a)^3} - \frac{4\sqrt{3}(S^{4/3} - aS^{2/3} + a^2)}{(S^{2/3}+a)^2\{3a^2 + (2S^{2/3}-a)^2\}}\right]. \tag{73}$$

Here the deceleration parameter has the value $q = -\frac{1}{3}$ as $m = \frac{2}{3}$. For these class of solutions, it is observed that the universe is accelerating.

4.7.1. $Model\ I : (\xi = \xi_0)$

When $n = 0$, Eq. (17) reduces to $\xi = \xi_0$ = constant and hence Eq. (73) with Eq. (16) gives

$$\mu = \frac{3D\xi_0}{\gamma S^{2/3}} - \frac{1}{8\pi\alpha\gamma}\left[\frac{S^{4/3} - aS^{2/3} + a^2}{(S^{2/3}+a)^2} exp\{-6\ arctan(\frac{2S^{2/3}-a}{\sqrt{3}a})\}\right]^{D^2/Ka^2} \times \left[kS^{-2} + K + \frac{2D^2}{3Ka}(3kS^{-2}+K)\left[\frac{(S^{2/3}+a)^2}{S^{4/3} - aS^{2/3} + a^2}\right]\right] \times \left[\frac{S^{2/3}-a}{(S^{2/3}+a)^3} - \frac{4\sqrt{3}(S^{4/3} - aS^{2/3} + a^2)}{(S^{2/3}+a)^2\{3a^2 + (2S^{2/3}-a)^2\}}\right]. \tag{74}$$

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4.7.2. $Model\ II : (\xi = \xi_0\mu)$

When $n = 0$, Eq. 17) reduces to $\xi = \xi_0\mu$ and hence Eq. (73) with Eq. (16) gives

$$\mu = -\frac{1}{8\pi\alpha(\gamma - 3D\xi_0 S^{-2/3})}\left[\frac{S^{4/3} - aS^{2/3} + a^2}{(S^{2/3}+a)^2} exp\{-6\ arctan(\frac{2S^{2/3}-a}{\sqrt{3}a})\}\right]^{D^2/Ka^2} \times \left[kS^{-2} + K + \frac{2D^2}{3Ka}(3kS^{-2}+K)\left[\frac{(S^{2/3}+a)^2}{S^{4/3} - aS^{2/3} + a^2}\right]\right] \times$$

$$\left[\frac{S^{2/3}-a}{(S^{2/3}+a)^2}-\frac{4\sqrt{3}(S^{4/3}-aS^{2/3}+a^2)}{(S^{2/3}+a)^2\{3a^2+(2S^{2/3}-a)^2\}}\right]. \tag{75}$$

In the above we have presented a number of new solutions to the Einstein field equations with variable cosmological constant and gravitational constant which satisfy the Hubble variation law given by Eq. (8). It remarkable that this simple law leads to a wide class of solutions. It is interesting to observe that solutions are admitted in which the gravitational constant may be increasing with time (cf. Abdel-Rahman, 1990). The anstaz utilised to solve the Einstein field equations (3)-(4) is very simple. It might be worthwhile to investigate other possibilities that lead to solutions to the Einstein's field equations with interesting behaviour of the gravitational constant and cosmological constant.

5. Conclusions

In this paper we have investigated Einstein's equation in the presence of a viscous fluid, for the Robertson-Walker universe within the framework of general relativity, where the gravitational constant G and the cosmological parameter Λ are variables. We utilize a form for the Hubble parameter ($H = DS^{-m}$) to generate a number of solutions to Einstein field equations. For these class of solutions where $m = 0, -2, \frac{1}{2}, \frac{2}{3}$, we find the decelerating parameters as negative. These class of solutions are consistent with the recent observations of supernovae Ia [20,21] which require the present universe is accelerating. From our results we observe that μ is decreasing with time with suitable choice of constants whereas G is increasing function of time. The possibility of an increasing G has been suggested by Abdel-Rahaman [32], Arbab [63] and Massa [64].

Assuming an *ad hoc* law of the form $\xi(t) = \xi_0\mu^n$, where μ is the energy density and n is the positive index, we have obtained exact solutions. The models discussed here are isotropic and homogeneous and, in view of the assumption of isotropy the sheer viscosity is absent. The effect of the bulk viscosity is to produce a change in the perfect fluid. We observe that Murphy's conclusion [53] about the absence of big bang type singularity in the finite past in models with bulk viscous fluid is, in general, not true.

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